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Multi-interval dissipative Sturm–Liouville boundary-value problems with distributional coefficients

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The paper investigates spectral properties of multi-interval Sturm–Liouville operators with distributional coefficients. Constructive descriptions of all self-adjoint and maximal dissipative/accumulative extensions in terms of boundary conditions are given. Sufficient conditions for the resolvents of these operators to be operators of the trace class and for the systems of root functions to be complete are found. The results are new for one-interval boundary-value problems as well.

Keywords: *Sturm–Liouville operator; multi-interval boundary value problems; distributional coefficients; maximal dissipative extension; completeness of root functions.*

1. Introduction. Differential operators, generated by the Sturm—Liouville expression

$$l(y) = -(py')' + qy,$$

arise in numerous problems of analysis and its applications. The classical assumptions on its coefficients are the following:

$$q \in C([a,b];\mathbb{R}), \quad 0 < p \in C^1([a,b];\mathbb{R}).$$

Principal statements of the theory of such operators remain true under more general assumptions

$$q, 1/p \in L_1([a,b],\mathbb{C}).$$

However, many problems of mathematical physics require the study of differential operators with complex coefficients which are Radon measures or even more singular distributions. In papers

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[1—4], a new approach to the investigation of such operators was proposed based on the definition of these operators as *quasi*-differential, which also allows one to consider differential operators of higher order [3, 5].

The purpose of this paper is to develop a spectral theory of not self-adjoint Sturm—Liouville operators given on a finite system of bounded intervals under minimal conditions for the regularity of the coefficients.

Multi-interval differential and quasi-differential operators were investigated, particularly, in papers [7—9].

2. Preliminary results. Let $[a,b]$ be a compact interval, $m\in\mathbb{N}$, and let $a=a_0<a_1<\ldots<a_m=b$ be a partition of the interval $[a,b]$ into m parts. Let us consider the space $L_2([a,b],\mathbb{C})$ as a direct sum $\oplus_{k=1}^m L_2([a_{k-1},a_k],\mathbb{C})$ which consists of vector functions $f=\oplus_{k=1}^m f_k$ such that $f_k\in L_2([a_{k-1},a_k],\mathbb{C})$.

Let, on each interval (a_{k-1},a_k), $k\in\{1,\ldots,m\}$, the formal Sturm-Liouville differential expression

$$l_k(y)=-(p_k(t)y')'+q_k(t)y+i((r_k(t)y)'+r_k(t)y'), \tag{1}$$

be given with coefficients p_k, q_k, and r_k which satisfy the conditions:

$$q_k=Q_k', \quad \frac{1}{\sqrt{|p_k|}}, \frac{Q_k}{\sqrt{|p_k|}}, \frac{r_k}{\sqrt{|p_k|}}\in L_2([a_{k-1},a_k],\mathbb{C}), \tag{2}$$

where the derivatives Q_k' are understood in the sense of distributions.

Similarly to [3] (see also [1, 4]), we introduce the quasi-derivatives by the coefficients of expression (1) on each interval $[a_{k-1},a_k]$ in the following way:

$$D_k^{[0]}y:=y;$$

$$D_k^{[1]}y:=p_k y'-(Q_k+ir_k)y;$$

$$D_k^{[2]}y:=(D_k^{[1]}y)'+\frac{Q_k-ir_k}{p_k}D_k^{[1]}y+\frac{Q_k^2+r_k^2}{p_k}y.$$

We also denote, for all $t\in[a_{k-1},a_k]$: $\hat{y}_k(t)=(D_k^{[0]}y(t),D_k^{[1]}y(t))\in\mathbb{C}^2$.

Under assumptions (2), these expressions are Shin—Zettl quasi-derivatives (see [11, 10]). One can easily verify that, for the smooth coefficients p_k, q_k, and r_k, the equality $l_k(y)=-D_k^{[2]}y$ holds.

Therefore, one may correctly define expressions (1) under assumptions (2) as Shin—Zettl quasi-differential expressions:

$$l_k[y]:=-D_k^{[2]}y.$$

The corresponding Shin—Zettl matrices (see [10, 11]) have the form

$$A_k = \begin{pmatrix} \dfrac{Q+ir}{p} & \dfrac{1}{p} \\ -\dfrac{Q^2+r^2}{p} & -\dfrac{Q-ir}{p} \end{pmatrix} \in L_1([a,b];\mathbb{C}^{2\times 2}). \tag{3}$$

Then, on the Hilbert spaces $L_2((a_{k-1}, a_k), \mathbb{C})$, the minimal and maximal differential operators are defined, which are generated by the quasi-differential expressions $l_k[y]$ (see [10, 11]):

$$L_{k,1} : y \to l_k[y], \quad \mathrm{Dom}(L_{k,1}) := \{y \in L_2 \,|\, y, D_k^{[1]}y \in AC([a_{k-1}, a_k], \mathbb{C}), D_k^{[2]}y \in L_2\},$$

$$L_{k,0} : y \to l_k[y], \quad \mathrm{Dom}(L_{k,0}) := \{y \in \mathrm{Dom}(L_{k,1}) \,|\, \hat{y}_k(a_{k-1}) = \hat{y}_k(a_k) = 0.\}.$$

Results of [10, 11] for general Shin—Zettl quasi-differential operators together with formula (3) imply that the operators $L_{k,1}$, $L_{k,0}$ are closed and densely defined on the space $L_2([a_{k-1}, a_k], \mathbb{C})$.

In the case where p_k, q_k, and r_k are real-valued, the operator $L_{k,0}$ is symmetric with the deficiency index (2, 2), $L^*_{k,0} = L_{k,1}$, and $L^*_{k,1} = L_{k,0}$.

3. Dissipative boundary-value problems. We consider the space $L_2([a,b], \mathbb{C})$ as a direct sum $\oplus_{k=1}^{m} L_2([a_{k-1}, a_k], \mathbb{C})$ which consists of vector functions $f = \oplus_{i=1}^{m} f_i$ such that $f_i \in L_2([a_{i-1}, a_i], \mathbb{C})$. In this space $L_2([a,b], \mathbb{C})$, we consider the maximal and minimal operators $L_{\max} = \oplus_{i=1}^{m} L_{i,1}$ and $L_{\min} = \oplus_{i=1}^{m} L_{i,0}$.

It is easy to see that the operators $L_{\max}$, $L_{\min}$ are closed and densely defined on the space $L_2([a,b], \mathbb{C})$.

Throughout the rest of the paper, we assume that (the) functions p_k, q_k, and r_k are *real-valued* for all k, and, therefore, the operators $L_{k,0}$ are symmetric with the deficiency indices (2, 2). Then the operator $L_{\min}$ is symmetric with the deficiency index $(2m, 2m)$ and $L^*_{\min} = L_{\max}$, $L^*_{\max} = L_{\min}$.

Then the problem of describing all its self-adjoint, maximal dissipative and maximal accumulative extensions in terms of homogeneous boundary conditions of the canonical form naturally arises. For this purpose, it is convenient to apply the approach based on the concept of boundary triplets. It was developed in the papers by Kochubei [12], see also book [13] and references therein.

Note that the minimal operator $L_{\min}$ may be not semi-bounded even in the case of a single-interval boundary-value problem since the function p may reverse sign.

Recall that a *boundary triplet* of a closed densely defined symmetric operator T with equal (finite or infinite) deficiency indices is called a triplet (H, Γ_1, Γ_2), where H is an auxiliary Hilbert space, and Γ_1, Γ_2 are the linear maps from $\mathrm{Dom}(T^*)$ into H such that:

1. for any $f, g \in \mathrm{Dom}(T^*)$, there holds

$$(T^* f, g)_{\mathcal{H}} - (f, T^* g)_{\mathcal{H}} = (\Gamma_1 f, \Gamma_2 g)_H - (\Gamma_2 f, \Gamma_1 g)_H,$$

2. for any $g_1, g_2 \in H$, there is a vector $f \in \mathrm{Dom}(T^*)$ such that $\Gamma_1 f = g_1$ and $\Gamma_2 f = g_2$.

The definition of the boundary triplet implies that $f \in \mathrm{Dom}(T)$, iff $\Gamma_1 f = \Gamma_2 f = 0$. A boundary triplet (H, Γ_1, Γ_2) with $\dim H = n$ exists for any symmetric operator T with equal non-zero deficiency indices (n, n) $(n \leqslant \infty)$, but it is not unique.

For the minimal quasi-differential operators $L_{k,0}$, the boundary triplet is explicitly given by the following theorem which follows from the results of [2].

Theorem 1. *For every* $k=1,\ldots,m$ *, the triplet* $(\mathbb{C}^2,\Gamma_{1,k},\Gamma_{2,k})$ *, where* $\Gamma_{1,k},\Gamma_{2,k}$ *are linear maps*

$$\Gamma_{1,k}y:=(D_k^{[1]}y(a_{k-1}+),-D_k^{[1]}y(a_k-)),\quad \Gamma_{2,k}y:=(y(a_{k-1}+),y(a_k-)),$$

from $\mathrm{Dom}(L_{k,1})$ *onto* $\mathbb{C}^2$ *is a boundary triplet for the operator* $L_{k,0}$.

For the minimal operator $L_{\min}$ in the space $L_2([a,b],\mathbb{C})$, the boundary triplet is explicitly given by the following theorem.

Theorem 2. *The triplet* $(\mathbb{C}^{2m},\Gamma_1,\Gamma_2)$ *, where* Γ_1,Γ_2 *are linear maps*

$$\Gamma_1 y:=(\Gamma_{1,1}y,\Gamma_{1,2}y,\ldots,\Gamma_{1,m}y),\quad \Gamma_2 y:=(\Gamma_{2,1}y,\Gamma_{2,2}y,\ldots,\Gamma_{2,m}y),\tag{4}$$

from $\mathrm{Dom}(L_{\max})$ *onto* $\mathbb{C}^{2m}$ *is a boundary triplet for the operator* $L_{\min}$.

Denote, by L_K , the restriction of operator $L_{\max}$ onto the set of functions $\mathrm{y}\in\mathrm{Dom}(L_{\max})$ satisfying the homogeneous boundary condition

$$(K-I)\Gamma_1 y+i(K+I)\Gamma_2 y=0,\tag{5}$$

where K is an arbitrary bounded operator on the space $\mathbb{C}^{2m}$.

Similarly, denote by L^K , the restriction of $L_{\max}$ onto the set of functions $\mathrm{y}\in\mathrm{Dom}(L_{\max})$ satisfying the homogeneous boundary condition

$$(K-I)\Gamma_1 y-i(K+I)\Gamma_2 y=0,\tag{6}$$

where K is an arbitrary bounded operator on the space $\mathbb{C}^{2m}$.

Theorem 1 and [13, Th. 1.6] lead to the following description of all self-adjoint, maximal dissipative and maximal accumulative extensions of operator $L_{\max}$.

Theorem 3. *Every* L_K *with* K *being a contracting operator in the space* $\mathbb{C}^{2m}$ *, is a maximal dissipative extension of the operator* $L_{\min}$*. Similarly, every* L^K *with* K *being a contracting operator in* $\mathbb{C}^{2m}$ *is a maximal accumulative extension of the operator* $L_{\min}$ *. Conversely, for any maximal dissipative (respectively, maximal accumulative) extension* $\tilde{L}$ *of the operator* $L_{\min}$ *, there exists the unique contracting operator* K *such that* $\tilde{L}=L_K$ *(respectively,* $\tilde{L}=L^K$ *).*

The extensions L_K *and* L^K *are self-adjoint, iff* K *is a unitary operator on* $\mathbb{C}^{2m}$ *.*

The mappings $K\to L_K$ *and* $K\to L^K$ *are injective.*

All functions from $\mathrm{Dom}(L_{\max})$ together with their first quasi-derivatives belong to $\oplus_{k=1}^m AC([a_{k-1},a_k],\mathbb{C})$. This implies that the following definition is correct.

Denote, by $\mathbf{f(t-)}$, the left germ and, by $\mathbf{f(t+)}$, the right germ of the continuous function f at a point t. Similarly to paper [2], we say that the boundary conditions which define the operator $L\subset L_{\max}$ are called *local*, if, for any functions $\mathrm{y}\in\mathrm{Dom}(L)$ and for any $y_1,\ldots,y_m\in\mathrm{Dom}(L_{\max})$, the equalities $\mathbf{y_j(a_j-)=y(a_j-)}$, $\mathbf{y_j(a_j+)=y(a_j+)}$ and $\mathbf{y_j(a_k-)=y_j(a_k+)}=0$, $k\neq j$ imply that $\mathrm{y}_j\in\mathrm{Dom}(L)$. For $j=0$ and $j=m$, we consider only the right and left germs, respectively.

The following statement gives a description of the extensions L_K and L^K which are given by local boundary conditions.

Theorem 4. *The boundary conditions* (5) *and* (6) *defining the extensions* L_K *and* L^K, *respectively, are local, iff the matrix* K *has the block form*

$$K = \begin{pmatrix} K_{a_0} & 0 & \dots & 0 \\ 0 & K_{a_1} & \dots & 0 \\ 0 & 0 & \dots & K_{a_n} \end{pmatrix}, \tag{7}$$

where K_{a_1} *and* $K_{a_n} \in \mathbb{C}$ *and other* $K_{a_j} \in \mathbb{C}^{2\times 2}$.

4. Generalized resolvents. Let us recall that a *generalized resolvent* of a closed symmetric operator L in a Hilbert space $\mathcal{H}$ is an operator-valued function $\lambda \mapsto R_\lambda$ defined on $\mathbb{C}\setminus\mathbb{R}$ which can be represented as

$$R_\lambda f = P^+(L^+ - \lambda I^+)^{-1} f, \quad f \in \mathcal{H},$$

where L^+ is a self-adjoint extension of the operator L which acts in a certain Hilbert space $\mathcal{H}^+ \supset \mathcal{H}$, I^+ is the identity operator on $\mathcal{H}^+$, and P^+ is the orthogonal projection operator from $\mathcal{H}^+$ onto $\mathcal{H}$. It is known that an operator-valued function R_λ is a generalized resolvent of a symmetric operator L, iff it can be represented as

$$(R_\lambda f, g)_{\mathcal{H}} = \int_{-\infty}^{+\infty} \frac{d(F_\mu f, g)}{\mu - \lambda}, \quad f, g \in \mathcal{H},$$

where F_μ is a generalized spectral function of the operator L. This implies that the operator-valued function F_μ has the following properties:

1. For $\mu_2 > \mu_1$, the difference $F_{\mu_2} - F_{\mu_1}$ is a bounded non-negative operator.
2. $F_{\mu+} = F_\mu$ for any real μ.
3. For any $x \in \mathcal{H}$, the following equalities hold:

$$\lim_{\mu\to-\infty} \| F_\mu x \|_{\mathcal{H}} = 0, \quad \lim_{\mu\to+\infty} \| F_\mu x - x \|_{\mathcal{H}} = 0.$$

The following theorem provides a parametric description of all generalized resolvents of the symmetric operator $L_{\min}$ (see also [14]).

Theorem 5. 1) *Every generalized resolvent* R_λ *of the operator* $L_{\min}$ *in the half-plane* $\operatorname{Im}\lambda < 0$ *acts by the rule* $R_\lambda h = y$, *where* y *is a solution of the boundary-value problem*

$$l(y) = \lambda y + h,$$

$$(K(\lambda) - I)\Gamma_1 f + i(K(\lambda) + I)\Gamma_2 f = 0.$$

Here, $h(x) \in L_2([a,b], \mathbb{C})$ *and* $K(\lambda)$ *is a* $2m\times 2m$ *matrix-valued function which is holomorphic in the lower half-plane and such that* $\| K(\lambda) \| \leqslant 1$.

2) *In the half-plane* $\operatorname{Im}\lambda > 0$, *every generalized resolvent of operator* $L_{\min}$ *acts by* $R_\lambda h = y$, *where* y *is a solution of the boundary-value problem*

$$l(y) = \lambda y + h,$$

$$(K(\lambda)-I)\Gamma_1 f - i(K(\lambda)+I)\Gamma_2 f = 0.$$

Here, $h(x)\in L_2([a,b],\mathbb{C})$ *, and* $K(\lambda)$ *is a* $2m\times 2m$ *matrix-valued function which is holomorphic in the upper half-plane and satisfies* $\|K(\lambda)\|\leqslant 1$*. This parametrization of the generalized resolvents by the matrix-valued functions* $K(\lambda)$ *is bijective.*

5. Completeness of the system of root vectors. Results of paper [15] imply that, in the single-interval case under the assumptions made and additionally for $r_k = r \equiv 0$, the resolvents of the operators L_K and L^K are Hilbert—Schmidt operators. This result is amplified and refined by the following theorem.

Theorem 6. 1) *The resolvents of the maximal dissipative (maximal accumulative) operators* L_K *and* L^K *are Hilbert—Schmidt operators.*

2) *Let* $\delta > 0$ *exist such that, for any* $k\in\{1,2,\ldots,m\}$,

$$\left\{\frac{1}{p_k}, \frac{Q_k + ir_k}{p_k}\right\} \subset W_2^{\delta}([a_{k-1}, a_k], \mathbb{C}).$$

Then the resolvent of the maximal dissipative (maximal accumulative) operator L_K (L^K) *is an operator from the trace class, and its system of root functions is complete in the Hilbert space* $L_2([a,b],\mathbb{C})$.

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БАГАТОІНТЕРВАЛЬНІ ДИСИПАТИВНІ КРАЙОВІ ЗАДАЧІ ШТУРМА—ЛІУВІЛЛЯ З КОЕФІЦІЄНТАМИ-РОЗПОДІЛАМИ

Досліджено спектральні властивості багатоінтервальних операторів Штурма—Ліувілля з узагальненими функціями в коефіцієнтах. Дано конструктивний опис усіх самоспряжених, максимальних дисипативних/акумулятивних розширень мінімального оператора в термінах крайових умов. Знайдено достатні умови ядерності резольвент цих операторів та повноти систем їх кореневих функцій. Результати роботи є новими і для одноінтервальних крайових задач.

Ключові слова: *оператор Штурма—Ліувілля, багатоінтервальна крайова задача, сингулярні коефіцієнти, максимальне дисипативне розширення, повнота крайових функцій.*