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Derivations and automorphisms of locally matrix algebras and groups

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We describe derivations and automorphisms of infinite tensor products of matrix algebras. Using this description, we show that, for a countable–dimensional locally matrix algebra A over a field F, the dimension of the Lie algebra of outer derivations of A and the order of the group of outer automorphisms of A are both equal to $|F|^{\aleph_0}$, *where* $|F|$ *is the cardinality of the field F.*

Let A be the group of invertible elements of a unital locally matrix algebra A. We describe isomorphisms of groups [A*, A*]. In particular, we show that inductive limits of groups SLn(F) are determined by their Steinitz numbers.*

Keywords: *locally matrix algebra, derivation, automorphism.*

Let F be a ground field. Following [1], we call an associative F-algebra A a *locally matrix algebra,* if, for each finite subset of A, there exists a subalgebra $B \subset A$ containing this subset such that B is isomorphic to some matrix algebra $M_n(F)$ for $n \geqslant 1$. We call a locally matrix algebra A *unital,* if it contains a unit 1.

Let N be the set of all positive integers, and let P be the set of all primes. An infinite formal product of the form $s = \prod_{p \in P} p^{r_p}$, where $r_p \in N \cup \{0, \infty\}$ for all $p \in P$, is called *Steinitz number* (see [2]).

J.G. Glimm [3] proved that every countable-dimensional unital locally matrix algebra is uniquely determined by its Steinitz number. In [4, 5], we showed that this is no longer true for unital locally matrix algebras of uncountable dimensions.

S.A. Ayupov and K.K. Kudaybergenov [6] constructed an outer derivation of the countable-dimensional unital locally matrix algebra of Steinitz number 2^∞ and used it as an example of an outer derivation in a von Neumann regular simple algebra. In [7], H. Strade studied derivations of locally finite-dimensional locally simple Lie algebras over a field of characteristic 0.

Recall that a linear map $d : A \to A$ is called a *derivation*, if $d(xy) = d(x)y + x\,d(y)$ for arbitrary elements x, y from A.

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For an element $a \in A$, the adjoint operator $\mathrm{ad}_A(a) : A \to A$, $x \to [a, x]$, is an *inner derivation* of the algebra A.

Let $\mathrm{Der}(A)$ be the Lie algebra of all derivations of the algebra A, and let $\mathrm{Inder}(A)$ be the ideal of all inner derivations. The factor algebra $\mathrm{Outder}(A) = \mathrm{Der}(A)/\mathrm{Inder}(A)$ is called the algebra of *outer derivations* of A.

Let $\mathrm{Aut}(A)$ and $\mathrm{Inn}(A)$ be the group of automorphisms and the group of inner automorphisms of the algebra A, respectively. The factor group $\mathrm{Out}(A) = \mathrm{Aut}(A)/\mathrm{Inn}(A)$ is called the group of *outer automorphisms* of A.

Along with automorphisms of the algebra A, we consider the semigroup $P(A)$ of injective endomorphisms (embeddings) of A, $\mathrm{Aut}(A) \subseteq P(A)$.

The set $\mathrm{Map}(A, A)$ of all mappings $A \to A$ is equipped with the Tykhonoff topology (see [8]).

Theorem 1. *Let A be a locally matrix algebra.*

1) *The ideal* $\mathrm{Inder}(A)$ *is dense in* $\mathrm{Der}(A)$ *in the Tykhonoff topology*.

2) *Let the algebra A contain* 1. *Then the completion of* $\mathrm{Inn}(A)$ in $\mathrm{Map}(A, A)$ *in the Tykhonoff topology is the semigroup $P(A)$. In particular*, $\mathrm{Inn}(A)$ *is dense in* $\mathrm{Aut}(A)$.

G. Köthe [9] proved that every countable-dimensional unital locally matrix algebra is isomorphic to a tensor product of matrix algebras.

We describe derivations of infinite tensor products of matrix algebras.

Let I be an infinite set, and let $\mathbf{P}$ be a system of nonempty finite subsets of I. We say that the system $\mathbf{P}$ is *sparse*, if:

1) for any $S \in \mathbf{P}$, all nonempty subsets of S also lie in $\mathbf{P}$,

2) an arbitrary element $i \in I$ lies in no more than finitely many subsets from $\mathbf{P}$.

Let $\mathbf{A} = \bigotimes_{i \in I} A_i$ and let all algebras A_i be isomorphic to finite-dimensional matrix algebras over F. For a subset $S = \{i_1, \ldots, i_r\} \subset I$, the subalgebra $A_S := A_{i_1} \otimes \cdots \otimes A_{i_r}$ is a tensor factor of the algebra $\mathbf{A}$.

Let $\mathbf{P}$ be a system of nonempty finite subsets of I. Let f_S, $S \in \mathbf{P}$, be a system of linear operators $A \to A$. The sum $\sum_{S \in \mathbf{P}} f_S$ converges in the Tykhonoff topology if for an arbitrary element $a \in \mathbf{A}$ the set $\{S \in \mathbf{P} \mid f_S(a) \neq 0\}$ is finite. In this case, the operator $a \to \sum_{S \in \mathbf{P}} f_S(a)$ is a linear operator. Moreover, if every summand f_S is a derivation of the algebra $\mathbf{A}$, then this sum is also a derivation of the algebra $\mathbf{A}$.

Let $\mathbf{P}$ be a sparse system. For each subset $S \in \mathbf{P}$, we choose an element $a_S \in A_S$. The sum $\sum_{S \in \mathbf{P}} \mathrm{ad}_{\mathbf{A}}(a_S)$ converges in the Tykhonoff topology to a derivation of $\mathbf{A}$. Indeed, choose an arbitrary element $a \in \mathbf{A}$. Let $a \in A_{i_1} \otimes \cdots \otimes A_{i_r}$. Because of the sparsity of the system $\mathbf{P}$, for all but finitely many subsets $S \in \mathbf{P}$, we have $\{i_1, \ldots, i_r\} \cap S = \varnothing$, and therefore $\mathrm{ad}_{\mathbf{A}}(a_S)(a) = 0$. Let $D_{\mathbf{P}}$ be the vector space of all such sums, $D_{\mathbf{P}} \subseteq \mathrm{Der}(\mathbf{A})$.

For each algebra A_i, $i \in I$, choose a subspace A_i^0 such that $A_i = F \cdot 1_{A_i} \dot{+} A_i^0$ is a direct sum and 1_{A_i} is a unit element of A_i. Let E_i be a basis of A_i^0. For a subset $S = \{i_1, \ldots, i_r\}$ of the set I let $E_S := E_{i_1} \otimes \cdots \otimes E_{i_r} = \{a_1 \otimes \cdots \otimes a_r \mid a_k \in E_{i_k}, 1 \leqslant k \leqslant r\}$ and $\mathrm{ad}_{\mathbf{A}}(E_S) := \{\mathrm{ad}_{\mathbf{A}}(e) \mid e \in E_S\}$.

A description of derivations of the algebra $\mathbf{A}$ is given by the following theorem.

Theorem 2. 1) *Suppose that the set I is countable. Then* $\mathrm{Der}(\mathbf{A}) = \bigcup_{\mathbf{P}} D_{\mathbf{P}}$, *where the union is taken over all sparse systems of subsets of I.*

2) *Let I be an infinite (not necessarily countable) set. Let* **P** *be a sparse system of subsets of I. Then the union of finite sets of operators* $\bigcup_{S\in\mathbf{P}} \mathrm{ad}_{\mathbf{A}}(E_S)$ *is a topological basis of* $D_{\mathbf{P}}$.

Using this description, we prove the analog of the result of H. Strade [7] for locally matrix algebras.

Theorem 3. *Let A be a countable-dimensional locally matrix algebra. Then the Lie algebra* Outder(A) *is not locally finite-dimensional.*

We describe automorphisms and unital injective endomorphisms of a countable-dimensional unital locally matrix algebra A. We note that by the result of A.G. Kurosh ([1, Theorem 10]), the semigroup $P(A)$ of unital injective homomorphisms is strictly bigger than Aut(A).

The starting point here is again Köthe's theorem [9] stating that every countable-dimensional unital locally matrix algebra A is isomorphic to a countable tensor product of matrix algebras. Therefore $A \cong \bigotimes_{i=1}^{\infty} A_i,\ A_i \cong M_{n_i}(F),\ n_i \geqslant 1.$

Let $H_n,\ n_i \geqslant 1,$ be the subgroup of the group Inn(A) generated by conjugations by invertible elements from $\bigotimes_{i\geqslant n} A_i$. Clearly, $H_n \cong \mathrm{Inn}(\bigotimes_{i\geqslant n} A_i)$ and $\mathrm{Inn}(A) = H_1 > H_2 > \cdots$. For each $n \geqslant 1$, choose a system of representatives of left cosets $hH_{n+1},\ h \in H_n$, and denote it as X_n. We assume that each X_n contains the identical automorphism.

For an arbitrary sequence of automorphisms $\varphi_n \in X_n,\ n \geqslant 1$, the infinite product $\varphi = \varphi_1\varphi_2\cdots$ converges in the Tykhonoff topology. Clearly, $\varphi \in P(A)$.

Theorem 4. *An arbitrary unital injective endomorphism* $\varphi \in P(A)$ *can be uniquely represented as* $\varphi = \varphi_1\varphi_2\cdots$, *where* $\varphi_n \in X_n$ *for each* $n \geqslant 1$.

We call a sequence of automorphisms $\varphi_n \in H_n,\ n \geqslant 1,$ *integrable*, if, for an arbitrary element $a \in A$, the subspace spanned by all elements $\varphi_n\varphi_{n-1}\cdots\varphi_1(a),\ n \geqslant 1,$ is finite-dimensional.

Theorem 5. *An injective endomorphism* $\varphi = \varphi_1\varphi_2\cdots$, *where* $\varphi_n \in H_n,\ n \geqslant 1$, *is an automorphism, if and only if the sequence* $\{\varphi_n^{-1}\}_{n\geqslant 1}$ *is integrable.*

Using Theorems 3, 4, we determine dimensions of Lie algebras Der(A) and Outder(A) and orders of groups Aut(A) and Out(A), where A is a countable-dimensional locally matrix algebra.

We denote the cardinality of a set X as $|X|$. For two sets X and Y, let Map(Y, X) denote the set of all mappings from Y to X. Given two cardinals α, β and sets X, Y such that $|X| = \alpha$, $|Y| = \beta$ we define $\alpha^{\beta} = |\mathrm{Map}(Y,X)|$. As always $\aleph_0$ stands for the countable cardinality.

Theorem 6. *Let* $\mathbf{A} = \bigotimes_{i\in I} A_i$, *where I is an infinite set, and each algebra* A_i *is isomorphic to a matrix algebra over a field F of the dimension* >1. *Then* $\dim_F \mathrm{Der}(A) = \dim_F \mathrm{Outder}(A) = |F|^{|I|}$.

Theorem 7. *Let A be a countable-dimensional locally matrix algebra over a field F. Then* $\dim_F \mathrm{Der}(A) = \dim_F \mathrm{Outder}(A) = |F|^{\aleph_0}$.

Theorem 8. *Let A be a countable—dimensional locally matrix algebra over a field F. Then* $|\mathrm{Aut}(A)| = |\mathrm{Out}(A)| = |F|^{\aleph_0}$.

Consider the algebra $M_N(F)$ of $N\times N$ matrices over the ground field F having finitely many nonzero elements in each column.

Following [10], we call an $N\times N$ matrix *periodic* (more precisely: n-periodic), if it is block-diagonal diag($a, a, \ldots$), where a is an $n\times n$ matrix.

Let $M_n^p(F)$ be the subalgebra of $M_N(F)$ that consists of all n-periodic matrices. Clearly, $M_n^p(F) \cong M_n(F)$.

Let s be a Steinitz number. Then $M_s^p(F) = \bigcup_{n \in N,\, n|s} M_n^p(F)$ is a subalgebra of $M_N(F)$ (see [10]). By the Theorem of J. Glimm [3], $M_s^p(F)$ is the only (up to isomorphism) unital locally matrix algebra of Steinitz number s.

Let $GL_n^p(F)$ be the group of invertible elements of $M_n^p(F)$, $SL_n^p(F) = [GL_n^p(F), GL_n^p(F)]$. Clearly, $GL_n^p(F) \cong GL_n(F)$, $SL_n^p(F) \cong SL_n(F)$.

Let $n_1, n_2, \ldots$ be a sequence of positive integers such that $n_i \mid n_{i+1}$, $i \geqslant 1$, and let s be the least common multiple of the numbers $(n_i, i \geq 1)$. Then

$$GL_{n_1}^p(F) \subset GL_{n_2}^p(F) \subset \cdots, \ \bigcup_{i \geqslant 1} GL_{n_i}^p(F) = GL_s^p(F),$$

$$SL_{n_1}^p(F) \subset SL_{n_2}^p(F) \subset \cdots, \ \bigcup_{i \geqslant 1} SL_{n_i}^p(F) = SL_s^p(F).$$

Our aim is to describe isomorphisms between groups $SL_s^p(F)$. We will do it in a more general context of unital locally matrix algebras.

Recall that, for an arbitrary associative unital F-algebra R and an arbitrary positive integer $n \geqslant 2$, the elementary linear group $E_n(R)$ is the group generated by all transvections $t_{ij}(a) = I_n + e_{ij}(a)$, $1 \leqslant i \neq j \leqslant n$, where I_n is the identity $n \times n$ matrix, $a \in R$, $e_{ij}(a)$ is the $n \times n$ matrix having the element a at the (i,j)-position and zero elsewhere. Denote, by R^*, the group of invertible elements of algebra R.

Let A be an infinite-dimensional unital locally matrix algebra. Let a subalgebra $1 \in B \subset A$ be isomorphic to some matrix algebra $M_n(F)$ for $n \geqslant 4$ and let C be a centralizer of the subalgebra B in A. By the theorem of H.M. Wedderbun (see [11]), $A \cong M_n(C)$. We show that, in this case, $[A^*, A^*] \cong E_n(C)$. After that, it is sufficient to apply the description of isomorphisms of elementary linear groups over rings due to I.Z. Golubchik and A.V. Mikhalev [12, 13] and E.I. Zelmanov [14] in order to prove the following theorems.

Theorem 9. *Let A, B be unital locally matrix algebras. If the groups* $[A^*, A^*]$ *and* $[B^*, B^*]$ *are isomorphic, then the rings A and B are isomorphic or anti-isomorphic. Moreover, for any isomorphism* $\varphi : [A^*, A^*] \to [B^*, B^*]$, *either there exists a ring isomorphism* $\theta_1 : A \to B$ *such that* φ *is the restriction of* θ_1 *to* $[A^*, A^*]$ *or there exists a ring anti-isomorphism* $\theta_2 : A \to B$ *such that, for an arbitrary element* $g \in [A^*, A^*]$, *we have* $\varphi(g) = \theta_2(g^{-1})$.

If the algebras A, B are countable-dimensional, then Theorem 9 can be strengthened. In this case, without loss of generality, we assume that $A = M_s^p(F)$, where s is the Steinitz number of the algebra A. The algebra $M_s^p(F)$ is closed with respect to the transposition $t : M_s^p(F) \to M_s^p(F)$, $g \to g^t$, which is an anti-isomorphism.

Theorem 10. *Let A, B be countable-dimensional unital locally matrix algebras. If the groups* $[A^*, A^*]$ *and* $[B^*, B^*]$ *are isomorphic, then the F-algebras A and B are isomorphic. Moreover, an arbitrary isomorphism* $\varphi : [A^*, A^*] \to [B^*, B^*]$ *either extends to a ring isomorphism* $\theta_1 : A \to B$ *or there exists a ring isomorphism* $\theta_2 : A \to B$ *such that* $\varphi(g) = \theta_2((g^{-1})^t)$ *for all elements* $g \in [A^*, A^*]$.

Corollary. *Let* s_1, s_2 *be Steinitz numbers. Then* $SL_{s_1}^p(F) \cong SL_{s_2}^p(F)$, *if and only if* $s_1 = s_2$.

REFERENCES

1. Kurosh, A. (1942). Direct decompositions of simple rings. Rec. Math. [Mat. Sbornik] N.S., **11**, No. 3, pp. 245-264.
2. Steinitz, E. (1910). Algebraische Theorie der Körper. J. Reine Angew. Math., **137**, pp. 167-309.
3. Glimm, J.G. (1960). On a certain class of operator algebras. Trans. Amer. Math. Soc., **95**, No. 2, pp. 318-340.
4. Bezushchak, O. & Oliynyk, B. (2020). Unital locally matrix algebras and Steinitz numbers. J. Algebra Appl. https://doi.org/10.1142/S0219498820501807
5. Bezushchak, O. & Oliynyk, B. (2020). Primary decompositions of unital locally matrix algebras. Bull. Math. Sci., **10**, No. 1. https://doi.org/10.1142/S166436072050006X
6. Ayupov, S. & Kudaybergenov, K. (2020). Infinite dimensional central simple regular algebras with outer derivations. Lobachevskii J. Math., **41**, No. 3, pp. 326-332. https://doi.org/10.1134/S1995080220030063
7. Strade, H. (1999). Locally finite dimensional Lie algebras and their derivation algebras. Abh. Math. Sem. Univ. Hamburg, **69**, pp. 373-391. https://doi.org/10.1007/BF02940886
8. Willard, S. (2004). General Topology. Mineola, New York: Dover Publications.
9. Köthe, G. (1931). Schiefkörper unendlichen Ranges uber dem Zentrum. Math. Ann., **105**, pp. 15-39.
10. Bezushchak, O.O. & Sushchans'kyi, V.I. (2016). Groups of periodically defined linear transformations of an infinite-dimensional vector space. Ukr. Math. J., **67**, No. 10, pp. 1457-1468. https://doi.org/10.1007/s11253-016-1165-x
11. Drozd, Yu.A. & Kirichenko, V.V. (1994). Finite dimensional algebras. Berlin, Heidelberg, New York: Springer.
12. Golubchik, I.Z. & Mikhalev, A.V. (1983). Isomorphisms of the general linear group over an associative ring. Vestn. Mosk. Univ. Ser. 1 Mat. Mekh., No. 3, pp. 61-72.
13. Golubchik, I.Z. (1998). Linear groups over associative rings (Unpublished Doctor thesis). Ufa Scientific Center, Bashkir State Pedagogical Institute, Ufa, Russia (in Russian).
14. Zelmanov, E.I. (1985). Isomorphisms of linear groups over associative rings. Sib. Mat. Zh., **26**, No. 4, pp. 49-67.

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ДИФЕРЕНЦІЮВАННЯ ТА АВТОМОРФІЗМИ
ЛОКАЛЬНО МАТРИЧНИХ АЛГЕБР І ГРУП

Описано диференціювання та автоморфізми нескінченних тензорних добутків матричних алгебр. З використанням цього опису показано, що для зліченновимірної локально матричної алгебри A над полем F розмірності алгебри Лі зовнішніх диференціювань A і порядок групи зовнішніх автоморфізмів A збігаються і дорівнюють $|F|^{\aleph_0}$, де $|F|$ означає потужність поля F.

Нехай A^* — група оборотних елементів унітальної локально матричної алгебри A. Описано ізоморфізми групи $[A^*, A^*]$. Зокрема, показано, що індуктивні границі груп $SL_n(F)$ визначаються їх числами Стейніца.

Ключові слова: *локально матрична алгебра, диференціювання, автоморфізм.*