Let $F$ be a ground field. Following [1], we call an associative $F$-algebra $A$ a locally matrix algebra, if, for each finite subset of $A$, there exists a subalgebra $B \subset A$ containing this subset such that $B$ is isomorphic to some matrix algebra $M_n(F)$ for $n \geq 1$. We call a locally matrix algebra $A$ unital, if it contains a unit 1.

Let $N$ be the set of all positive integers, and let $P$ be the set of all primes. An infinite formal product of the form $s = \prod_{p \in P} p^{r_p}$, where $r_p \in N \cup \{0, \infty\}$ for all $p \in P$, is called Steinitz number (see [2]).

J.G. Glimm [3] proved that every countable-dimensional unital locally matrix algebra is uniquely determined by its Steinitz number. In [4, 5], we showed that this is no longer true for unital locally matrix algebras of uncountable dimensions.

S.A. Ayupov and K.K. Kudaybergenov [6] constructed an outer derivation of the countable-dimensional unital locally matrix algebra of Steinitz number $2^\omega$ and used it as an example of an outer derivation in a von Neumann regular simple algebra. In [7], H. Strade studied derivations of locally finite-dimensional locally simple Lie algebras over a field of characteristic 0.

Recall that a linear map $d : A \to A$ is called a derivation, if $d(xy) = d(x)y + x d(y)$ for arbitrary elements $x, y$ from $A$. 

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For an element \( a \in A \), the adjoint operator \( \text{ad}_A(a) : A \to A, x \to [a, x] \), is an inner derivation of the algebra \( A \).

Let \( \text{Der}(A) \) be the Lie algebra of all derivations of the algebra \( A \), and let \( \text{Inder}(A) \) be the ideal of all inner derivations. The factor algebra \( \text{Outer}(A) = \text{Der}(A) / \text{Inder}(A) \) is called the algebra of outer derivations of \( A \).

Let \( \text{Aut}(A) \) and \( \text{Inn}(A) \) be the group of automorphisms and the group of inner automorphisms of the algebra \( A \), respectively. The factor group \( \text{Out}(A) = \text{Aut}(A) / \text{Inn}(A) \) is called the group of outer automorphisms of \( A \).

Along with automorphisms of the algebra \( A \), we consider the semigroup \( P(A) \) of injective endomorphisms (embeddings) of \( A \), \( \text{Aut}(A) \subseteq P(A) \).

The set \( \text{Map}(A, A) \) of all mappings \( A \to A \) is equipped with the Tykhonoff topology (see [8]).

**Theorem 1.** Let \( A \) be a locally matrix algebra.

1) The ideal \( \text{Inder}(A) \) is dense in \( \text{Der}(A) \) in the Tykhonoff topology.

2) Let the algebra \( A \) contain 1. Then the completion of \( \text{Inn}(A) \) in \( \text{Map}(A, A) \) is dense in \( \text{Aut}(A) \).

G. Köthe [9] proved that every countable-dimensional unital locally matrix algebra is isomorphic to a tensor product of matrix algebras.

We describe derivations of infinite tensor products of matrix algebras.

Let \( I \) be an infinite set, and let \( P \) be a system of nonempty finite subsets of \( I \). We say that the system \( P \) is sparse, if:

1) for any \( S \in P \), all nonempty subsets of \( S \) also lie in \( P \),

2) an arbitrary element \( i \in I \) lies in no more than finitely many subsets from \( P \).

Let \( A = \bigotimes_{i \in I} A_i \) and let all algebras \( A_i \) be isomorphic to finite-dimensional matrix algebras over \( F \). For a subset \( S = \{i_1, \ldots, i_r\} \subset I \), the subalgebra \( A_S := A_{i_1} \otimes \cdots \otimes A_{i_r} \) is a tensor factor of the algebra \( A \).

Let \( P \) be a system of nonempty finite subsets of \( I \). Let \( f_S, S \in P \), be a system of linear operators \( A \to A \). The sum \( \sum_{S \in P} f_S \) converges in the Tykhonoff topology if for an arbitrary element \( a \in A \) the set \( \{S \in P \mid f_S(a) \neq 0\} \) is finite. In this case, the operator \( a \to \sum_{S \in P} f_S(a) \) is a linear operator. Moreover, if every summand \( f_S \) is a derivation of the algebra \( A \), then this sum is also a derivation of the algebra \( A \).

Let \( P \) be a sparse system. For each subset \( S \in P \), we choose an element \( a_S \in A_S \). The sum \( \sum_{S \in P} \text{ad}_A(a_S) \) converges in the Tykhonoff topology to a derivation of \( A \). Indeed, choose an arbitrary element \( a \in A \). Let \( a \in A_{i_1} \otimes \cdots \otimes A_{i_r} \). Because of the sparsity of the system \( P \), for all but finitely many subsets \( S \in P \), we have \( \{i_1, \ldots, i_r\} \cap S = \emptyset \), and therefore \( \text{ad}_A(a_S)(a) = 0 \). Let \( D_P \) be the vector space of all such sums, \( D_P \subseteq \text{Der}(A) \).

For each algebra \( A_i, i \in I \), choose a subspace \( A_i^0 \) such that \( A_i = F \cdot 1_A_i + A_i^0 \) is a direct sum and \( 1_A_i \) is a unit element of \( A_i \). Let \( E_i \) be a basis of \( A_i^0 \). For a subset \( S = \{i_1, \ldots, i_r\} \) of the set \( I \) let \( E_S := E_{i_1} \otimes \cdots \otimes E_{i_r} = \{a_S \in A_S \mid a_{i_k} \in E_{i_k}, 1 \leq k \leq r\} \) and \( \text{ad}_A(E_S) := \{\text{ad}_A(e) \mid e \in E_S\} \).

A description of derivations of the algebra \( A \) is given by the following theorem.

**Theorem 2.** 1) Suppose that the set \( I \) is countable. Then \( \text{Der}(A) = \bigcup_{P} D_P \), where the union is taken over all sparse systems of subsets of \( I \).
2) Let $I$ be an infinite (not necessarily countable) set. Let $P$ be a sparse system of subsets of $I$. Then the union of finite sets of operators $\bigcup_{S \in P} \text{ad}_A(E_S)$ is a topological basis of $D_P$.

Using this description, we prove the analog of the result of H. Strade [7] for locally matrix algebras.

**Theorem 3.** Let $A$ be a countable-dimensional locally matrix algebra. Then the Lie algebra $\text{Outder}(A)$ is not locally finite-dimensional.

We describe automorphisms and unital injective endomorphisms of a countable-dimensional unital locally matrix algebra $A$. We note that by the result of A.G. Kurosh ([1, Theorem 10]), the semigroup $P(A)$ of unital injective homomorphisms is strictly bigger than $\text{Aut}(A)$.

The starting point here is again Köthe’s theorem [9] stating that every countable-dimensional unital locally matrix algebra $A$ is isomorphic to a countable tensor product of matrix algebras. Therefore $A \cong \bigotimes_{i=1}^{\infty} A_i$, $A_i \cong M_{n_i}(F)$, $n_i \geq 1$.

Let $H_n$, $n \geq 1$, be the subgroup of the group $\text{Inn}(A)$ generated by conjugations by invertible elements from $\bigotimes_{i \geq n} A_i$. Clearly, $H_n \cong \text{Inn}(\bigotimes_{i \geq n} A_i)$ and $\text{Inn}(A) = H_1 > H_2 > \ldots$. For each $n \geq 1$, choose a system of representatives of left cosets $hH_{n+1}$, $h \in H_n$, and denote it as $X_n$. We assume that each $X_n$ contains the identical automorphism.

For an arbitrary sequence of automorphisms $\phi_n \in X_n$, $n \geq 1$, the infinite product $\phi = \phi_1 \phi_2 \ldots$ converges in the Tychonoff topology. Clearly, $\phi \in P(A)$.

**Theorem 4.** An arbitrary unital injective endomorphism $\phi \in P(A)$ can be uniquely represented as $\phi = \phi_1 \phi_2 \ldots$, where $\phi_n \in X_n$, for each $n \geq 1$.

We call a sequence of automorphisms $\phi_n \in H_n$, $n \geq 1$, integrable, if, for an arbitrary element $a \in A$, the subspace spanned by all elements $\phi_n \phi_{n-1} \cdots \phi_1(a)$, $n \geq 1$, is finite-dimensional.

**Theorem 5.** An injective endomorphism $\phi = \phi_1 \phi_2 \ldots$, where $\phi_n \in H_n$, $n \geq 1$, is an automorphism, if and only if the sequence $\{\phi_n^{-1}\}_{n \geq 1}$ is integrable.

Using Theorems 3, 4, we determine dimensions of Lie algebras $\text{Der}(A)$ and $\text{Outder}(A)$ and orders of groups $\text{Aut}(A)$ and $\text{Out}(A)$, where $A$ is a countable-dimensional locally matrix algebra.

We denote the cardinality of a set $X$ as $|X|$. For two sets $X$ and $Y$, let $\text{Map}(Y, X)$ denote the set of all mappings from $Y$ to $X$. Given two cardinals $\alpha$, $\beta$ and sets $X$, $Y$ such that $|X| = \alpha$, $|Y| = \beta$ we define $\alpha^\beta = |\text{Map}(Y, X)|$. As always $\aleph_0$ stands for the countable cardinality.

**Theorem 6.** Let $A = \bigotimes_{i=1}^{\infty} A_i$, where $I$ is an infinite set, and each algebra $A_i$ is isomorphic to a matrix algebra over a field $F$ of the dimension $\geq 1$. Then $\dim_F \text{Der}(A) = \dim_F \text{Out}(A) = |F|^{|I|}$.

**Theorem 7.** Let $A$ be a countable-dimensional locally matrix algebra over a field $F$. Then $\dim_F \text{Der}(A) = \dim_F \text{Out}(A) = |F|^{|\aleph_0|}$.

**Theorem 8.** Let $A$ be a countable—dimensional locally matrix algebra over a field $F$. Then $|\text{Aut}(A)| = |\text{Out}(A)| = |F|^{|\aleph_0|}$.

Consider the algebra $M_N(F)$ of $N \times N$ matrices over the ground field $F$ having finitely many nonzero elements in each column.

Following [10], we call an $N \times N$ matrix periodic (more precisely: $n$-periodic), if it is block-diagonal $\text{diag}(a, a, \ldots)$, where $a$ is an $n \times n$ matrix.

Let $M^p_n(F)$ be the subalgebra of $M_N(F)$ that consists of all $n$-periodic matrices. Clearly, $M^p_n(F) \equiv M_n(F)$.
Let $s$ be a Steinitz number. Then $M_s^p(F) = \bigcup_{n \in N, n \mid s} M_n^p(F)$ is a subalgebra of $M_s(F)$ (see [10]). By the Theorem of J. Glimm [3], $M_s^p(F)$ is the only (up to isomorphism) unital locally matrix algebra of Steinitz number $s$.

Let $GL_n^p(F)$ be the group of invertible elements of $M_n^p(F)$, $SL_n^p(F) = [GL_n^p(F), GL_n^p(F)]$. Clearly, $GL_n^p(F) \equiv GL_n(F)$, $SL_n^p(F) \equiv SL_n(F)$.

Let $n_1, n_2, \ldots$ be a sequence of positive integers such that $n_i \mid n_{i+1}$, $i \geq 1$, and let $s$ be the least common multiple of the numbers $(n_i, i \geq 1)$. Then

\[ GL_{n_1}^p(F) \subset GL_{n_2}^p(F) \subset \cdots, \bigcup_{i \geq 1} GL_{n_i}^p(F) = GL_s^p(F), \]

\[ SL_{n_1}^p(F) \subset SL_{n_2}^p(F) \subset \cdots, \bigcup_{i \geq 1} SL_{n_i}^p(F) = SL_s^p(F). \]

Our aim is to describe isomorphisms between groups $SL_s^p(F)$. We will do it in a more general context of unital locally matrix algebras.

Recall that, for an arbitrary associative unital $F$-algebra $R$ and an arbitrary positive integer $n \geq 2$, the elementary linear group $E_n(R)$ is the group generated by all transvections $t_{ij}(a) = I_n + e_{ij}(a)$, $1 \leq i \neq j \leq n$, where $I_n$ is the identity $n \times n$ matrix, $a \in R$, $e_{ij}(a)$ is the $n \times n$ matrix having the element $a$ at the $(ij)$-position and zero elsewhere. Denote, by $R^*$, the group of invertible elements of algebra $R$.

Let $A$ be an infinite-dimensional unital locally matrix algebra. Let a subalgebra $1 \in B \subseteq A$ be isomorphic to some matrix algebra $M_n(F)$ for $n \geq 4$ and let $C$ be a centralizer of the subalgebra $B$ in $A$. By the theorem of H.M. Wedderburn (see [11]), $A \equiv M_n(C)$. We show that, in this case, $[A^*, A^*] \equiv E_n(C)$. After that, it is sufficient to apply the description of isomorphisms of elementary linear groups over rings due to I.Z. Golubchik and A.V. Mikhalev [12, 13] and E.I. Zelmanov [14] in order to prove the following theorems.

**Theorem 9.** Let $A, B$ be unital locally matrix algebras. If the groups $[A^*, A^*]$ and $[B^*, B^*]$ are isomorphic, then the rings $A$ and $B$ are isomorphic or anti-isomorphic. Moreover, for any isomorphism $\phi : [A^*, A^*] \to [B^*, B^*]$, either there exists a ring isomorphism $\theta_1 : A \to B$ such that $\phi$ is the restriction of $\theta_1$ to $[A^*, A^*]$ or there exists a ring anti-isomorphism $\theta_2 : A \to B$ such that, for an arbitrary element $g \in [A^*, A^*]$, we have $\phi(g) = \theta_2(g^{-1})$.

If the algebras $A, B$ are countable-dimensional, then Theorem 9 can be strengthened. In this case, without loss of generality, we assume that $A = M_s^p(F)$, where $s$ is the Steinitz number of the algebra $A$. The algebra $M_s^p(F)$ is closed with respect to the transposition $t : M_s^p(F) \to M_s^p(F)$, $g \to g^t$, which is an anti-isomorphism.

**Theorem 10.** Let $A, B$ be countable-dimensional unital locally matrix algebras. If the groups $[A^*, A^*]$ and $[B^*, B^*]$ are isomorphic, then the $F$-algebras $A$ and $B$ are isomorphic. Moreover, an arbitrary isomorphism $\phi : [A^*, A^*] \to [B^*, B^*]$ either extends to a ring isomorphism $\theta_1 : A \to B$ or there exists a ring isomorphism $\theta_2 : A \to B$ such that $\phi(g) = \theta_2((g^{-1})^t)$ for all elements $g \in [A^*, A^*]$.

**Corollary.** Let $s_1, s_2$ be Steinitz numbers. Then $SL_{s_1}^p(F) \equiv SL_{s_2}^p(F)$, if and only if $s_1 = s_2$. 
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ДИФЕРЕНЦІЮВАННЯ ТА АВТОМОРФІЗМИ
ЛОКАЛЬНО МАТРИЧНИХ АЛГЕБР І ГРУП
Описано диференціювання та автоморфізми нескінчених тензорних добутків матричних алгебр. З використанням цього опису показано, що для зліченновимірної локально матричної алгебри $A$ над полем $F$ розмірність алгебри $L$ зовнішніх диференціювань $A$ і порядок групи зовнішніх автоморфізмів $A$ збігаються і дорівнюють $|F|_{\aleph_0}$, де $|F|$ означає потужність поля $F$.

Нехай $A^*$ — група оборотних елементів унітальної локально матричної алгебри $A$. Описано ізоморфізми групи $[A^*, A^*]$. Зокрема, показано, що індуктивні границі груп $SL_n(F)$ визначаються їх числами Стейніца.

Ключові слова: локально матрична алгебра, диференціювання, автоморфізм.