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Linear groups saturated by subgroups of finite central dimension

Presented by Academician of the NAS of Ukraine A.M. Samoilenko

Let F be a field, A be a vector space over F, and G be a subgroup of GL(F, A). We say that G has a dense family of subgroups having finite central dimension, if, for every pair of subgroups H, K of G such that $H \leqslant K$ and H is not maximal in K, there exists a subgroup L of finite central dimension such that $H \leqslant L \leqslant K$ (we can note that L can match with one of the subgroups H or K). We study locally solvable linear groups with a dense family of subgroups having finite central dimension.

Keywords: *linear group, infinite groups, infinite-dimensional linear group, dense family of subgroups, locally soluble groups, finite central dimension.*

We recall that a group G that is isomorphic to a group of automorphisms of a vector space A over a field F is called a linear group. We denote the group of all such automorphisms by GL(F, A). If $\dim_F(A)$, the dimension of A over F, is finite, n, then we say that G is a *finite-dimensional* linear group. It is well known that GL(F, A) can be identified with the group of $n \times n$ matrices with entries in F. From the outset, finite-dimensional linear groups have played an important role in group theory. This is partly due to the correspondence mentioned above, but also because of the rich interplay between geometric and algebraic ideas associated with such groups.

The study of the subgroups of GL(F, A) in the case where A is infinite-dimensional over F has been much more limited and normally requires some additional restrictions. There is quite a large array of papers that show the effectiveness of applying various natural limitations for the study of infinite-dimensional linear groups (see survey articles [1—4]). One area that proved to be quite effective was the study of linear groups that have a very big family of subgroups having finite central dimension.

Let G be a subgroup of $GL(F, A)$ and $Z = C_A(G)$. It is not hard to see that a subspace Z is G-invariant, and G acts trivially on Z. Therefore, we see that G actually acts on the quotient-space A/Z.

Let G be a subgroup of $GL(F, A)$. Then the *central dimension of* G is a dimension of the quotient-space $A/\zeta_G(A)$.

In particular, if $\dim_F(A/\zeta_G(A))$ is finite, then we will say that *G has finite central dimension*. According to the definition, linear groups having finite central dimension are quite close to finite-dimensional linear groups. Therefore, it was obvious to study the infinite-dimensional linear groups saturated by subgroups of finite central dimension. Among the works on this subject, we mention [5, 6, 1, 7—10].

Among the restrictions that play a significant role in the study of both finite and infinite groups, we can highlight the restriction associated with the presence of one or another family of dense subgroups in a group. Let P be a some property. We say that a group G has a dense family of subgroups having property P, if, for every pair of subgroups H, K of G such that $H \leqslant K$ and H is not maximal in K, there exists a subgroup L, having property P, such that $H \leqslant L \leqslant K$ (we note that L can match with one of the subgroups H or K).

Groups with different natural dense families have been considered by many authors (see, for example, [11—15]).

In this paper, we will apply this restriction to the study of infinite-dimensional linear groups.

Let F be a field, A be a vector space over F, and G be subgroup of $GL(F, A)$. We say that *G has a dense family of subgroups, having finite central dimension*, if, for every pair of subgroups H, K of G such that $H \leqslant K$ and H is not maximal in K, there exists a subgroup L of finite central dimension such that $H \leqslant L \leqslant K$ (we note that L can match with one of the subgroups H or K).

Note that infinite-dimensional linear groups, whose proper subgroups have finite central dimension, have this property. Locally soluble groups, whose proper subgroups have finite central dimension, were studied in paper [5]. Therefore, in this paper, the study of linear groups with a dense family of subgroups, having finite central dimension, will be conducted under the additional condition of their local solvability.

Lemma 1. *Let F be a field, A be a vector space over F, and G be subgroup of* $GL(F, A)$.

(i) *If H, K are two subgroups of G such that* $H \leqslant K$ *and a subgroup K has finite central dimension, then the subgroup H has finite central dimension.*

(ii) *If H, K are two subgroups having finite central dimension, then the both subgroups* $\langle H, K\rangle$ *and* $H \cap K$ *have finite central dimension.*

(iii) *If G has finite central dimension and* $\mathrm{char}(F) = p$ *is a prime, then G includes a normal elementary Abelian p-subgroup L such that G/L is isomorphic to some subgroup of* $GL(n, F)$, *where* $n = \dim_F(A/C_A(G))$.

(iv) *If G has finite central dimension and* $\mathrm{char}(F) = 0$, *then G includes a normal Abelian torsion-free subgroup L such that G/L is isomorphic to some subgroup of* $GL(n, F)$, *where* $n = \dim_F(A/C_A(G))$.

Corollary 1. *Let F be a field, A be a vector space over F, having infinite dimension, and G be a subgroup of* $GL(F, A)$. *If H, K are two subgroups of G such that* $H \leqslant K$, *and a subgroup H has infinite central dimension, then the subgroup K has infinite central dimension.*

Corollary 2. *Let F be a field, A be a vector space over F, having infinite dimension, and G be a subgroup of* $GL(F, A)$. *If G has a dense family of subgroups having finite central dimension, then every subgroup of G having infinite central dimension is maximal.*

Lemma 2. *Let F be a field, A be a vector space over F, having infinite dimension, and G be a subgroup of* $GL(F, A)$. *Suppose that G includes the subgroups B, K satisfying the following conditions:*

(i) *B is an infinite elementary Abelian p-subgroup, and K is a quasicyclic p-subgroup for some prime p;*

(ii) *B is K-invariant;*

(iii) $B \cap K = \langle 1 \rangle$.

If G has a dense family of subgroups having finite central dimension, then K has finite central dimension.

Corollary 3. *Let F be a field, A be a vector space over F, having infinite dimension, and G be a subgroup of GL(F, A). Suppose that G includes the subgroups B, K satisfying the following conditions:*

(i) *B is an infinite elementary Abelian p-subgroup, and K is a quasicyclic p-subgroup for some prime p;*

(ii) *B is K-invariant;*

(iii) $B \cap K$ *is finite.*

If G has a dense family of subgroups having finite central dimension, then K has finite central dimension.

Lemma 3. *Let F be a field, A be a vector space over F, having infinite dimension, and G be a subgroup of GL(F, A). Suppose that G includes the subgroups B, K satisfying the following conditions:*

(i) *B is an infinite elementary Abelian p-subgroup, and K is a quasicyclic q-subgroup, where p, q are primes and* $p \neq q$*;*

(ii) *B is K-invariant;*

If G has a dense family of subgroups having finite central dimension, then either K has finite central dimension or B is a minimal K-invariant subgroup.

Lemma 4. *Let F be a field, A be a vector space over F, having infinite dimension, and G be a locally soluble subgroup of GL(F, A). Suppose that G includes a subgroup K, having infinite central dimension. If G has a dense family of subgroups having finite central dimension, then K satisfies the following conditions:*

(i) *K is a quasicyclic or cyclic p-subgroup for some prime p;*

(ii) *every proper subgroup of K has finite central dimension;*

(iii) *K is a maximal subgroup of G.*

Theorem. *Let F be a field, A be a vector space over F, having infinite dimension, and G be a locally soluble subgroup of GL(F, A). Suppose that G has infinite central dimension. If G has a dense family of subgroups having finite central dimension, then G is a group of one of following types:*

(i) *G is a cyclic or quasicyclic p-group for some prime p;*

(ii) $G = K \times L$, *where K is a cyclic or quasicyclic p-group for some prime p, and L is a group of prime order;*

(iii) $G = \langle a, b \mid |a| = 2^n, |b| = 2, a^b = a^t, \text{where } t = 1 + 2^{n-1}, n \geqslant 3 \rangle$;

(iv) $G = \langle a, b \mid |a| = 2^n, |b| = 2, a^b = a^t, \text{where } t = -1 + 2^{n-1}, n \geqslant 3 \rangle$;

(v) $G = \langle a, b \mid |a| = 2^n, |b| = 2, a^b = a^{-1} \rangle$;

(vi) $G = \langle a, b \mid |a| = 2^n, b^2 = a^t, \text{where } t = 2^{n-1}, a^b = a^{-1} \rangle$;

(vii) $G = \langle a, b \mid |a| = p^n, |b| = p, a^b = a^t, \text{where } t = 1 + p^{n-1}, n \geqslant 2, p \text{ is an odd prime} \rangle$;

($viii$) $G = \langle a \rangle \rightthreetimes \langle b \rangle$, $|a| = p^n$ *where p is an odd prime,* $|b| = q$, *q is a prime,* $q \neq p$*;*

(ix) $G = B \rightthreetimes \langle a \rangle$, $|a| = p^n$, $B = C_G(B)$ *is an elementary Abelian q-subgroup, p and q are primes,* $p \neq q$, *B is a minimal normal subgroup of G;*

(*x*) *$G = K \rtimes \langle b\rangle$, where K is a quasicyclic 2-subgroup, $|b| = 2$ and $x^b = x^{-1}$ for each element $x \in K$;*

(*xi*) *$G = K\langle b\rangle$, where $K = \langle a_n \mid a_1{}^p = 1, a_{n+1}{}^p = a_n, n \in N\rangle$ is a quasicyclic 2-subgroup, $b^2 = a_1$ and $a_n{}^b = a_n{}^{-1}$, $n \geqslant 2$.*

(*xii*) *$G = K \rtimes \langle b\rangle$, where K is a quasicyclic p-subgroup, p is an odd prime, $K = C_G(K)$, $|b| = q$ is a prime such that $p \neq q$;*

(*xiii*) *$G = Q \rtimes K$, where K is a quasicyclic p-subgroup, $Q = C_G(Q)$ is an elementary Abelian q-subgroup, p, q are primes, $p \neq q$, Q is a minimal normal subgroup of G.*

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ЛІНІЙНІ ГРУПИ З НАСИЧЕНИМИ ПІДГРУПАМИ
СКІНЧЕННОЇ ЦЕНТРАЛЬНОЇ РОЗМІРНОСТІ

Нехай *F* – поле, *A* – векторний простір над *F*, *G* — підгрупа *GL*(*F*, *A*). Будемо говорити, що *G* має щільне сімейство підгруп, які мають скінченну центральну розмірність, якщо для кожної пари підгруп *H*, *K* з *G* такої, що $H \leqslant K$ і H немаксимальна в K, існує така підгрупа L скінченної центральної розмірності, що $H \leqslant L \leqslant K$ (зазначимо, що L може збігатися з однією з підгруп H або K). У роботі описані локально розв'язні лінійні групи з щільним сімейством підгруп, що мають скінченну центральну розмірність.

Ключові слова: *лінійна група, нескінченна група, нескінченно розмірна лінійна група, щільне сімейство підгруп, локально розв'язна група, скінченна центральна розмірність.*

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ЛИНЕЙНЫЕ ГРУППЫ С НАСЫЩЕННЫМИ ПОДГРУППАМИ КОНЕЧНОЙ ЦЕНТРАЛЬНОЙ РАЗМЕРНОСТИ

Пусть F — поле, A — векторное пространство над F, G — подгруппа $GL(F, A)$. Будем говорить, что G имеет плотное семейство подгрупп, имеющих конечную центральную размерность, если для каждой пары подгрупп H, K из G такой, что $H \leqslant K$ и H немаксимальная в K, существует такая подгруппа L конечной центральной размерности, что $H \leqslant L \leqslant K$ (отметим, что L может совпадать с одной из подгрупп H или K). В работе описаны локально разрешимые линейные группы с плотным семейством подгрупп, имеющих конечную центральную размерность.

Ключевые слова: *линейная группа, бесконечная группа, бесконечно размерная линейная группа, плотное семейство подгрупп, локально разрешимая группа, конечная центральная размерность.*