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On boundary-value problems for generalized analytic and harmonic functions

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The present paper is a natural continuation of our last articles on the Riemann, Hilbert, Dirichlet, Poincaré, and, in particular, Neumann boundary-value problems for quasiconformal, analytic, harmonic functions and the so-called A-harmonic functions with arbitrary boundary data that are measurable with respect to the logarithmic capacity. Here, we extend the corresponding results to generalized analytic functions $h: D \to \mathbb{C}$ *with sources* $g: \partial_{\bar{z}} h = g \in L^p$, $p > 2$, *and to generalized harmonic functions* U *with sources* $G: \Delta U = G \in L^p$, $p > 2$. *Our approach is based on the geometric (functional-theoretic) interpretation of boundary values in comparison with the classical operator approach in PDE. Here, we will establish the corresponding existence theorems for the Poincaré problem on directional derivatives and, in particular, for the Neumann problem to the Poisson equations* $\Delta U = G$ *with arbitrary boundary data that are measurable with respect to the logarithmic capacity. A few mixed boundary-value problems are considered as well. These results can be also applied to semilinear equations of mathematical physics in anisotropic and inhomogeneous media.*

Keywords: *Poincaré and Neumann boundary-value problems, generalized analytic functions, generalized harmonic functions, logarithmic capacity and potential.*

1. Introduction. Our last paper [1] was devoted to the proof of the existence of nonclassical solutions of the Riemann, Hilbert, and Dirichlet boundary-value problems with arbitrary measurable boundary data with respect to the logarithmic capacity for the equations

$$\partial_{\bar{z}} h(z) = g(z) \tag{1}$$

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with the real-valued function g in the class L^p, $p > 2$. We call continuous solutions h of Eq. (1) with the generalized first partial derivatives by *Sobolev generalized analytic functions with sources g*.

Recall that the research of the Dirichlet problem for harmonic functions in a unit disk $\mathbb{D} := \{z \in \mathbb{C} : |z| < 1\}$ with arbitrary measurable boundary data is due to the Luzin dissertation, see its original text [2] and its reprint [3]. The corresponding analogs of the Luzin theorems in terms of the logarithmic capacity in more general domains can be found in [4] (see also [5]). Later on, a series of results in various boundary-value problems have been formulated and proved in terms of the logarithmic capacity, see its definition and properties in [6]. Further, *q.e. means quasieverywhere* with respect to the logarithmic capacity.

The present paper contains, in particular, the proof of the existence of nonclassical solutions to the Poincaré problem on the directional derivatives and, in particular, to the Neumann problem with arbitrary measurable boundary data with respect to the logarithmic capacity for the Poisson equations

$$\Delta U(z) = G(z) \tag{2}$$

with real-valued functions G of a class $L^p(D)$, $p > 2$, in the corresponding domains $D \subset \mathbb{C}$. For short, we call continuous solutions to (2) of the class $W^{2,p}_{\text{loc}}(D)$ *generalized harmonic functions with the source* G. Note that, by the Sobolev embedding theorem, see Theorem I.10.2 in [7], such functions belong to the class C^1.

However, the paper is started and finished by nonlinear mixed boundary-value problems of mathematical physics.

In this connection, recall one more useful definition. Let D be a domain in $\mathbb{C}$ whose boundary consists of a finite collection of mutually disjoint Jordan curves. A family of mutually disjoint Jordan arcs $J_\zeta : [0,1] \to \overline{D}$, $\zeta \in \partial D$, with $J_\zeta([0,1)) \subset D$ and $J_\zeta(1) = \zeta$ that is continuous in the parameter ζ is called a *Bagemihl—Seidel system* or, in short, of *class* $\mathcal{BS}$ (see [8]). For the rest definitions, we refer to the paper [1].

2. On mixed boundary-value problems. Remark 3 in our previous paper [1] makes it possible to formulate a series of nonlinear boundary-value problems in terms of Bagemihl-Seidel systems for generalized analytic functions including mixed boundary-value problems. In order to demonstrate the potentiality of our approach, we give here a couple of results. Namely, arguing similarly to the proof of Theorem 1 in [1], see also Theorem 1.10 in [9], we obtain, for instance, by Theorem 10 and Lemma 5 in [10], the following statement on mixed boundary-value problems.

Theorem 1. *Let D be a domain in $\mathbb{C}$ whose boundary consists of a finite number of mutually disjoint Jordan curves, $\varphi : \partial D \times \mathbb{C} \to \mathbb{C}$, satisfy the Carathéodory conditions and let $\nu : \partial D \to \mathbb{C}$, $|\nu(\zeta)| \equiv 1$, be measurable with respect to the logarithmic capacity. Suppose also that $g : \mathbb{C} \to \mathbb{R}$ is in $C^\alpha(\mathbb{C})$, $\alpha \in (0,1)$, with compact support, $\{\gamma^+_\zeta\}_{\zeta \in \partial D}$ and $\{\gamma^-_\zeta\}_{\zeta \in \partial D}$ are families of Jordan arcs of the class $\mathcal{BS}$ in D and $\mathbb{C} \setminus \overline{D}$, correspondingly.*

Then there exist generalized analytic functions $f^+ : D \to \mathbb{C}$ and $f^- : \mathbb{C} \setminus \overline{D} \to \mathbb{C}$ with the source g such that

$$f^+(\zeta) = \varphi\left(\zeta, \left[\frac{\partial f}{\partial \nu}\right]^-(\zeta)\right) \qquad q.e. \text{ on } \partial D, \tag{3}$$

where $f^+(\zeta)$ *and* $\left[\frac{\partial f}{\partial \nu}\right]^-(\zeta)$ *are limits of the functions* $f^+(z)$ *and* $\frac{\partial f^-}{\partial \nu}(z)$ *as* $z \to \zeta$ *along* γ_ζ^+ *and* γ_ζ^-, *correspondingly.*

Furthermore, the space of all such couples (f^+, f^-) *has the infinite dimension for any such prescribed functions* g, φ, ν *and collections* γ_ζ^+ *and* γ_ζ^-, $\zeta \in \partial D$.

Theorem 1 is a special case of the following lemma on the mixed problem with shift.

Lemma 1. *Under the conditions of Theorem 1, let, in addition,* $\beta : \partial D \to \partial D$ *be a homeomorphism keeping components of* ∂D *such that* β *and* β^{-1} *have the Luzin* (N)*-property with respect to the logarithmic capacity.*

Then there exist generalized analytic functions $f^+ : D \to \mathbb{C}$ *and* $f^- : \overline{\mathbb{C}} \setminus \overline{D} \to \mathbb{C}$ *with the source* g *such that*

$$f^+(\beta(\zeta)) = \varphi\left(\zeta, \left[\frac{\partial f}{\partial \nu}\right]^-(\zeta)\right) \qquad \text{q.e. on } \partial D, \tag{4}$$

where $f^+(\zeta)$ *and* $\left[\frac{\partial f}{\partial \nu}\right]^-(\zeta)$ *are limits of the functions* $f^+(z)$ *and* $\frac{\partial f^-}{\partial \nu}(z)$ *as* $z \to \zeta$ *along* γ_ζ^+ *and* γ_ζ^-, *correspondingly.*

Furthermore, the space of all such couples (f^+, f^-) *has the infinite dimension for any such prescribed* g, φ, ν, β *and collections* $\{\gamma_\zeta^+\}_{\zeta \in \partial D}$ *and* $\{\gamma_\zeta^-\}_{\zeta \in \partial D}$.

Proof. Indeed, by relations (2.21) in [6] and Theorem 1.10 in [9], the logarithmic (Newtonian) potential $\mathcal{N}_G$ with the source $G = 2g$,

$$\mathcal{N}_G(z) := \frac{1}{2\pi} \int_{\mathbb{C}} \ln|z - w|\, G(w)\, dm(w), \tag{5}$$

is in $C^{2,\alpha}(\mathbb{C})$. Setting $P := \mathcal{N}_G$, we conclude by elementary calculations that the function

$$H(z) := \overline{\nabla P(z)}, \quad z \in \mathbb{C}, \quad \nabla P := P_x + iP_y, \quad z = x + iy, \tag{6}$$

is a generalized analytic function in the class $C^{1,\alpha}(\mathbb{C})$ with the source g. Hence, the function of the directional derivatives of $H(z)$ along the unit vectors $\nu(\zeta)$

$$h(\zeta) := \frac{\partial H}{\partial \nu}(\zeta), \quad \zeta \in \partial D, \tag{7}$$

as the projection of its gradient $\nabla H := H_x + iH_y$, $z = x + iy$, into $\nu(\zeta)$ is measurable with respect to the logarithmic capacity, because $\nabla H(z) \in C^\alpha(\mathbb{C})$.

Now, let $a : \partial D \to \mathbb{C}$ be an arbitrary function that is measurable with respect to the logarithmic capacity. Then, by Theorem 6 in [10], there exist analytic functions $\mathcal{A}^- : \mathbb{C} \setminus \overline{D} \to \mathbb{C}$ such that

$$\lim_{z \to \zeta} \frac{\partial \mathcal{A}^-}{\partial \nu}(z) = a(\zeta) \quad \text{q.e. on } \partial D. \tag{8}$$

Setting $f^- = H + \mathcal{A}^-$ on $\mathbb{C}\setminus\overline{D}$ and $\psi = h + a$ on ∂D, we see that the function $\psi:\partial D\to\mathbb{C}$ can be measurable with respect to the logarithmic capacity, f^- is a generalized analytic function with the source g in $\mathbb{C}\setminus\overline{D}$, and

$$\lim_{z\to\zeta}\frac{\partial f^-}{\partial\nu}(z) = \psi(\zeta) \quad \textit{q.e. on } \partial D. \tag{9}$$

Next, the function $\Psi(\zeta) := \varphi(\zeta, \psi(\zeta))$ is measurable with respect to the logarithmic capacity on ∂D (see Example 1 to Remark 4 in paper [1]). Then the function $\Phi = \Psi\circ\beta^{-1}$ is also measurable with respect to the logarithmic capacity, because the homeomorphism β has the Luzin (N)-property with respect to the logarithmic capacity.

Consequently, by Theorem 1 in [10], there exist analytic functions $\mathcal{A}^+ : D\to\mathbb{C}$ such that $\mathcal{A}^+(z)\to\Phi(\zeta) - H(\zeta)$ as $z\to\zeta$ along γ_ζ q.e. on ∂D. Setting $f^+ = H + \mathcal{A}^+$ on D, we see that f^+ is a generalized analytic function with the source g in D such that $f^+(z)\to\Phi(\zeta)$ as $z\to\zeta$ along γ_ζ q.e. on ∂D.

Thus, f^+ and f^- are the desired functions, because β^{-1} also has the Luzin (N)-property. It remains to note that the space of all such couples (f^+, f^-) has the infinite dimension, because the space of all functions $\psi:\partial D\to\mathbb{C}$ which are measurable with respect to the logarithmic capacity has the infinite dimension (see arguments in Remark 2 of paper [1]).

Remark 1. In the case of Jordan domains D, following the same scheme, namely, applying once more Theorem 6 in [10] instead of Theorem 1 in [10] in the final stage of the above proof, the similar statement can be derived for the boundary gluing conditions of the form

$$\left[\frac{\partial f^+}{\partial\nu_*}\right](\beta(\zeta)) = \varphi\left(\zeta, \left[\frac{\partial f^-}{\partial\nu}\right](\zeta)\right) \quad \textit{q.e. on } \partial D. \tag{10}$$

3. Poincaré and Neumann problems in terms of angular limits. In this section, we consider the Poincaré boundary-value problem on the directional derivatives and, in particular, the Neumann problem for the Poisson equations

$$\Delta U(z) = G(z) \tag{11}$$

with real-valued functions G of the classes $L^p(D)$ with $p > 2$ in the corresponding domains $D\subset\mathbb{C}$. Recall that a continuous solution U of (11) in the class $W^{2,p}_{\mathrm{loc}}$ is called a *generalized harmonic function with the source* G and that, by the Sobolev embedding theorem, such a solution belongs to the class C^1.

Theorem 2. *Let* D *be a Jordan domain with the quasihyperbolic boundary condition,* ∂D *have a tangent q.e.,* $\nu:\partial D\to\mathbb{C}, |\nu(\zeta)|\equiv 1$*, be in* $\mathcal{CBV}(\partial D)$*, and* $\varphi:\partial D\to\mathbb{R}$ *be measurable with respect to the logarithmic capacity.*

Suppose that $G: D\to\mathbb{R}$ *is in* $L^p(D)$*,* $p > 2$*. Then there exist generalized harmonic functions* $U: D\to\mathbb{R}$ *with the source* G *that have the angular limits*

$$\lim_{z\to\zeta}\frac{\partial U}{\partial\nu}(z) = \varphi(\zeta) \qquad \textit{q.e. on } \partial D. \tag{12}$$

Furthermore, the space of such functions U has the infinite dimension.

Proof. Indeed, let us extend the function G by zero outside of D, and let P be the logarithmic potential $\mathcal{N}_G$ with the source G, see (5). Then, by Lemma 3 in [6], $P \in W^{2,p}_{\mathrm{loc}}(\mathbb{C}) \cap C^{1,\alpha}_{\mathrm{loc}}(\mathbb{C})$ with $\alpha = (p-2)/p$ and $\Delta P = G$ a.e. in $\mathbb{C}$. Set

$$\varphi_*(\zeta) = \mathrm{Re}\,\nu(\zeta)\,H(\zeta), \quad \zeta \in \partial D\,, \tag{13}$$

where

$$H(z) := \overline{\nabla P(z)}, z \in \mathbb{C}, \quad \nabla P := P_x + iP_y, \quad z = x + iy\,. \tag{14}$$

Then, by Theorem 1 in [1], with $g = G/2$ in D and $\lambda = \overline{\nu}$ on ∂D, there exist generalized analytic functions h with the source g that have the angular limits

$$\lim_{z\to\zeta} \mathrm{Re}\,\nu(\zeta)\,h(z) = \varphi(\zeta) \qquad \textit{q.e. on } \partial D \tag{15}$$

and, moreover, by Remark 1, the given functions h can be represented in the form of the sums $\mathcal{A} + H$ with analytic functions $\mathcal{A}$ in D that have the angular limits

$$\lim_{z\to\zeta} \mathrm{Re}\,\nu(\zeta)\mathcal{A}(z) = \Phi(\zeta) \qquad \textit{q.e. on } \partial D\,. \tag{16}$$

With $\Phi(\zeta) := \varphi(\zeta) - \varphi_*(\zeta)$, $\zeta \in \partial D$, and the space of such analytic functions $\mathcal{A}$ has the infinite dimension.

Note that any indefinite integral $\mathcal{F}$ of such $\mathcal{A}$ in the simply connected domain D is also a single-valued analytic function, and the harmonic functions $u := \mathrm{Re}\,\mathcal{F}$ and $v := \mathrm{Im}\,\mathcal{F}$ satisfy the Cauchy--Riemann system $u_x = v_y$ and $u_y = -v_x$. Hence,

$$\mathcal{A} = \mathcal{F}' = \mathcal{F}_x = u_x + i\cdot v_x = u_x - i\cdot u_y = \overline{\nabla u}\,. \tag{17}$$

Consequently, setting $U_* = u + P$, we see that U_* is a generalized harmonic function with the source G and, moreover, by construction, $h = \overline{\nabla U_*}$.

Note also that the directional derivative of U_* along the unit vector ν is the projection of its gradient ΔU_* into ν, i.e., the scalar product of ν and ΔU_* interpreted as vectors in $\mathbb{R}^2$ and, consequently,

$$\frac{\partial U_*}{\partial \nu} = (\nu, \nabla U_*) = \mathrm{Re}\,\nu\cdot\overline{\nabla U_*} = \mathrm{Re}\,\nu\cdot h\,. \tag{18}$$

Thus, (15) implies (12) and the proof is complete.

Remark 2. We are able to say more in the case of $\mathrm{Re}\,n(\zeta)\overline{\nu(\zeta)} > 0$, where $n(\zeta)$ is the inner normal to ∂D at the point ζ. Indeed, the latter magnitude is a scalar product of $n = n(\zeta)$ and $\nu = \nu(\zeta)$ interpreted as vectors in $\mathbb{R}^2$, and it has the geometric sense of projection of the vector ν onto n. In view of (12), since the limit $\varphi(\zeta)$ is finite, there is a finite limit $U(\zeta)$ of $U(z)$ as $z \to \zeta$ in D along the straight line passing through the point ζ and being parallel to the vector ν, because, along this line,

$$U(z)=U(z_0)-\int_0^1 \frac{\partial U}{\partial \nu}(z_0+\tau(z-z_0))\,d\tau\,. \tag{19}$$

Thus, at each point with condition (12), there is the directional derivative

$$\frac{\partial U}{\partial \nu}(\zeta):=\lim_{t\to 0}\frac{U(\zeta+t\cdot\nu)-U(\zeta)}{t}=\varphi(\zeta)\,. \tag{20}$$

In particular, in the case of the Neumann problem, $\mathrm{Re}\, n(\zeta)\overline{\nu(\zeta)}\equiv 1>0$, where $n=n(\zeta)$ denotes the unit interior normal to ∂D at the point ζ, and we have, by Theorem 2 and Remark 2, the following significant result.

Corollary 1. *Let D be a Jordan domain in $\mathbb{C}$ with the quasihyperbolic boundary condition, the unit inner normal $n(\zeta)$, $\zeta\in\partial D$, belong to the class $\mathcal{CBV}(\partial D)$, and $\varphi:\partial D\to\mathbb{R}$ be measurable with respect to the logarithmic capacity.*

Suppose that $G:D\to\mathbb{R}$ is in $L^p(D)$, $p>2$. Then one can find generalized harmonic functions $U:D\to\mathbb{R}$ with the source G such that, q.e. on ∂D, there exist:

1) the finite limit along the normal $n(\zeta)$

$$U(\zeta):=\lim_{z\to\zeta}U(z)\,,$$

2) the normal derivative

$$\frac{\partial U}{\partial n}(\zeta):=\lim_{t\to 0}\frac{U(\zeta+t\cdot n(\zeta))-U(\zeta)}{t}=\varphi(\zeta)\,,$$

3) the angular limit

$$\lim_{z\to\zeta}\frac{\partial U}{\partial n}(z)=\frac{\partial U}{\partial n}(\zeta)\,.$$

Furthermore, the space of such functions U has the infinite dimension.

4. Poincaré and Neumann problems and Bagemihl—Seidel systems. Arguing similarly to the last section, we obtain, by Theorem 6 in [10], the following statement.

Theorem 3. *Let D be a Jordan domain in $\mathbb{C}$, $\nu:\partial D\to\mathbb{C}$, $|\nu(\zeta)|\equiv 1$, and $\varphi:\partial D\to\mathbb{C}$ be measurable functions with respect to the logarithmic capacity, and $\{\gamma_\zeta\}_{\zeta\in\partial D}$ be a family of Jordan arcs of the class $\mathcal{BS}$ in D.*

Suppose also that $G:D\to\mathbb{R}$ is in $L^p(D)$, $p>2$. Then there exist generalized harmonic functions $U:D\to\mathbb{C}$ with the source G that have the limits along γ_ζ

$$\lim_{z\to\zeta}\frac{\partial U}{\partial \nu}(z)=\varphi(\zeta) \qquad \text{q.e. on } \partial D\,. \tag{21}$$

Furthermore, the space of such functions U has the infinite dimension.

Remark 3. As follows from the proofs of Theorems 2 and 3, the generalized harmonic functions U with a source $G\in L^p$, $p>2$, satisfying the Poincaré boundary conditions can be represented in the form of the sums $\mathcal{N}_G+U_*$ of the logarithmic (Newtonian) potential $\mathcal{N}_G$ that is a generalized harmonic function with the source G and harmonic functions U_* satisfying the corresponding Poincaré boundary conditions.

5. On the Riemann—Poincaré type problems for the Poisson equations. Finally, arguing similarly to the proofs of Corollaries 9 and 10 in [10] and being based on Lemma 1 in [1] and Lemma 1 in Section 2, correspondingly, we obtain the following consequences.

Corollary 2. *Let* D *be a domain in* $\mathbb{C}$ *whose boundary consists of a finite number of mutually disjoint Jordan curves,* $B:\partial D\to\mathbb{R}$ *and* $C:\partial D\to\mathbb{R}$ *be functions that are measurable with respect to the logarithmic capacity, and* $\alpha:\partial D\to\partial D$ *be a homeomorphism keeping components of* ∂D *such that* α *and* α^{-1} *have the Luzin* (N)*-property with respect to the logarithmic capacity.*

Suppose that $G:\mathbb{C}\to\mathbb{R}$ *is in* $L^p(\mathbb{C})$, $p>2$, *with compact support,* $\{\gamma_\zeta^+\}_{\zeta\in\partial D}$, *and* $\{\gamma_\zeta^-\}_{\zeta\in\partial D}$ *are families of Jordan arcs of the class* $\mathcal{BS}$ *in* D *and* $\mathbb{C}\setminus\overline{D}$, *correspondingly. Then there exist generalized harmonic functions* $u^+:D\to\mathbb{R}$ *and* $u^-:\overline{\mathbb{C}}\setminus\overline{D}\to\mathbb{R}$ *with the source* G *such that*

$$u^+(\alpha(\zeta))=B(\zeta)\cdot u^-(\zeta)+C(\zeta) \qquad \text{q.e. on } \partial D \tag{22}$$

where $u^+(\zeta)$ *and* $u^-(\zeta)$ *are limits of* $u^+(z)$ *and* $u^-(z)$ *az* $z\to\zeta$ *along* γ_ζ^+ *and* γ_ζ^-, *correspondingly.*

Furthermore, the space of all such couples (u^+,u^-) *has the infinite dimension for any such prescribed G, B, C,* α *and collections* $\{\gamma_\zeta^+\}_{\zeta\in\partial D}$ *and* $\{\gamma_\zeta^-\}_{\zeta\in\partial D}$.

In particular, we are able to obtain, from the following corollary, solutions of the problem on gluing of the Dirichlet problem in the unit disk $\mathbb{D}$ and the Neumann problem outside of $\mathbb{D}$ in the class of generalized harmonic functions with the source g.

Corollary 3. *Let* D *be a domain in* $\mathbb{C}$ *whose boundary consists of a finite number of mutually disjoint Jordan curves,* $\nu:\partial D\to\mathbb{C}$, $|\nu(\zeta)|\equiv 1$, *be a measurable function,* $\beta:\partial D\to\partial D$ *be a homeomorphism such that* β *and* β^{-1} *have the Luzin* (N)*-property, and* $\varphi:\partial D\times\mathbb{R}\to\mathbb{R}$ *satisfy the Carathéodory conditions with respect to the logarithmic capacity.*

Suppose that $G:\mathbb{C}\to\mathbb{R}$ *is in* C^α, $\alpha\in(0,1)$, *with compact support,* $\{\gamma_\zeta^+\}_{\zeta\in\partial D}$ *and* $\{\gamma_\zeta^-\}_{\zeta\in\partial D}$ *are families of Jordan arcs of the class* $\mathcal{BS}$ *in* D *and* $\mathbb{C}\setminus\overline{D}$, *correspondingly. Then there exist generalized harmonic functions* $u^+:D\to\mathbb{R}$ *and* $u^-:\overline{\mathbb{C}}\setminus\overline{D}\to\mathbb{R}$ *with the source* G *such that*

$$u^+(\beta(\zeta))=\varphi\left(\zeta,\left[\frac{\partial u}{\partial\nu}\right]^-(\zeta)\right) \qquad \text{q.e. on } \partial D, \tag{23}$$

where $u^+(\zeta)$ *and* $\left[\frac{\partial u}{\partial\nu}\right]^-(\zeta)$ *are limits of the functions* $u^+(z)$ *and* $\frac{\partial u^-}{\partial\nu}(z)$ *as* $z\to\zeta$ *along* γ_ζ^+ *and* γ_ζ^-, *correspondingly.*

Furthermore, the space of all such couples (u^+,u^-) *has the infinite dimension for any such prescribed G,* ν, β, φ *and collections* γ_ζ^+ *and* γ_ζ^-, $\zeta\in\partial D$.

The corresponding results on the boundary-value problems for semilinear equations of mathematical physics in anisotropic and inhomogeneous media with arbitrary measurable data can be proved on the basis of our factorization theorem in paper [11], cf. also [12].

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ПРО КРАЙОВІ ЗАДАЧІ ДЛЯ УЗАГАЛЬНЕНИХ АНАЛІТИЧНИХ ТА ГАРМОНІЧНИХ ФУНКЦІЙ

Робота є продовженням досліджень крайових задач Рімана, Гільберта, Діріхле, Пуанкаре і, зокрема, Неймана, для квазіконформних, аналітичних, гармонічних і так званих *A*-гармонічних функцій із довільними граничними даними, які є вимірюваними відносно логарифмічної ємності. Тут відповідні результати поширено на узагальнені аналітичні функції $h : D \to \mathbb{C}$ з джерелом $g : \partial_{\bar{z}} h = g \in L^p$, $p > 2$, і на узагальнені гармонічні функції U з джерелом $G : \Delta U = G \in L^p$, $p > 2$. Даний підхід заснований на геометричній (теоретико-функціональній) інтерпретації крайових задач у порівнянні з класичним операторним підходом у теорії РЧП. Встановлені відповідні теореми існування для задачі Пуанкаре для похідної за напрямком і, зокрема, для задачі Неймана для рівняння Пуассона $\Delta U = G$ з довільними граничними даними, що є вимірюваними відносно логарифмічної ємності. Також розглянуто декілька змішаних граничних задач. Ці результати можуть буть також застосовані до напівлінійних рівнянь математичної фізики в анізотропних та неоднорідних середовищах.

Ключові слова: *крайові задачі Пуанкаре і Неймана, узагальнені аналітичні функції, логарифмічна ємність, узагальнені гармонічні функції, логарифмічний потенціал.*